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GYROSCOPIC WAVES IN A ROTATING LIQUID LAYER

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The dispersion characteristics of gyroscopic waves in an incompressible liquid layer in a cavity of a rapidly rotating cylinder are studied. It is shown that in a viscous incompressible liquid layer, an inertial wave can be represented as the sum of six helical harmonics. The effects of the liquid viscosity and the ratio of the wave frequency to the angular velocity of rotation of the cylinder on the real and imaginary parts of the wavenumber are studied.

Key words: *rotating liquid, rotating cylinder, gyroscopic waves, dispersion characteristics.*

Introduction. Wave processes in a liquid with the effect of the Earth's rotation taken into account were considered in papers dealing with waves in the ocean (see, for example, [1–4] and the bibliography therein). The strong effect of rotation on the liquid dynamics is exemplified by rotary systems. Rotors with a liquid-containing cavity, in which the centrifugal force can be several hundred thousand or even several millions of times the gravity. Disturbance of the balance between the pressure gradient and the Coriolis force can lead to the generation of wave motions in the rotating liquid, which are called inertial or gyroscopic waves. These waves play an important role in the problems of dynamics of rotors, turbines, separators, centrifuges, and rotating aircraft containing a liquid, and in some geophysical problems (flows in the Earth's liquid core [5]). Wave phenomena in a rotating liquid layer can have a significant effect on a number of technological processes (in particular, sedimentation processes), phase equilibria in multicomponent liquids, and on the aircraft flight trajectory. The dynamics of inertial waves in a liquid filling a rotating cylinder has been experimentally studied in a number of papers (see [6, 7] and references therein). Stationary gyroscopic waves in a circular cylindrical layer of an ideal liquid bounded by solid walls were studied in [8, 9]. Dispersion relations were obtained and dependences of dimensionless wavelengths on dimensionless frequencies were constructed. The behavior of a low-viscosity liquid in a rotating horizontal cylinder as a function of the rotation velocity and the degree of filling was studied in [10]. The resonant generation of waves in a liquid filling a rotor cavity, which is the main factor responsible for instability of steady-state rotation of rotary systems was investigated in [11–19].

The present investigation of the properties of gyroscopic waves in a rotating liquid layer is motivated by interest in the problem of the stability of rotary systems with a fixed point at which the angular velocity of rotation of a rotor is maintained constant by a drive. A stability analysis method for similar rotary systems was proposed in [15, 17]. In this method, one of the main stages involves calculation of the moments of the forces exerted by the liquid on the rotor walls during steady-state conical precession of the rotor. It is easy to show that, in the case of conical precession, the steady-state hydrodynamic problem is directly related to the problem of generation of inertial waves in a liquid layer in a cavity of a solid body rotating around a stationary axis. In the present paper, we consider the kinematic part of the hydrodynamic problem, namely, the dispersion properties of gyroscopic waves in a rotating liquid layer.

1. Helical Harmonics. We consider an incompressible viscous liquid layer in an infinite cylinder of circular cross section, which rotates rapidly around the symmetry axis at a constant angular velocity Ω. The influence of gravity is ignored. We introduce a noninertial cylindrical coordinate system $Or\varphi z$ attached to the cylinder with

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the z axis directed along the symmetry axis. In the unperturbed state, the liquid moves together with the rotating solid cylindrical shell as a whole unit:

$$\boldsymbol{v}_0 = 0, \qquad p_0 = p_a + \rho\Omega^2(r^2 - b^2)/2 \tag{1.1}$$

(p_a is the pressure in the cylinder cavity free from the liquid, b is the radius of the free unperturbed surface, and ρ is the density).

After linearization (1.1), the motion of the liquid in the vicinity of the unperturbed state is described by linear Navier–Stokes equations and the incompressibility equations

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla \frac{p}{\rho} + 2[\boldsymbol{v}, \boldsymbol{\Omega}] + \nu\Delta\boldsymbol{v}, \qquad \operatorname{div}\boldsymbol{v} = 0. \tag{1.2}$$

The following boundary conditions are specified: the condition of attachment of liquid particles to the side wall of the cavity ($r = a$)

$$\boldsymbol{v} = 0, \tag{1.3}$$

the condition of zero tangential stresses, the continuity condition for normal stresses, and the kinematic condition on the free surface of the liquid ($r = b$)

$$\begin{gathered} \frac{1}{b}\,\frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial r} - \frac{v}{b} = 0, \qquad \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} = 0, \\ -p - \rho\Omega^2 bh + 2\mu\,\frac{\partial u}{\partial r} = 0, \qquad \frac{\partial h}{\partial t} - u = 0. \end{gathered} \tag{1.4}$$

Here u, v, and w are the radial, azimuthal, and axial components of the velocity $\boldsymbol{v}$, μ and $\nu = \mu/\rho$ are the dynamic and kinematic viscosities, respectively; the function $h(\varphi)$ defines the free-surface shape of the liquid: $r = b + h(\varphi)$ and p is the pressure.

Equations (1.2) admit particular solutions of the form

$$\varkappa\boldsymbol{v} = \operatorname{rot}\boldsymbol{v}; \tag{1.5}$$

$$\boldsymbol{v} = \boldsymbol{v}_*(r)\,\mathrm{e}^{i(\omega t + kz + m\varphi)}, \qquad m = 0, \pm 1, \pm 2, \ldots; \tag{1.6}$$

$$p = -2\rho\Omega\varkappa^{-1}w, \tag{1.7}$$

if the cyclic frequency ω, the vorticity parameter $\varkappa$, and k are linked by the relation

$$\omega\varkappa - 2\Omega k - i\nu\varkappa^3 = 0. \tag{1.8}$$

This is easily obtained by applying the rot operation to the first equation of (1.2) and using (1.5). For the helical fields studied, the first equation of (1.2) can be written as

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla\Big(\frac{p}{\rho} + \frac{2}{\varkappa}\,(\boldsymbol{v}, \boldsymbol{\Omega})\Big) + \operatorname{rot}\Big(\frac{2}{\varkappa}\,[\boldsymbol{v}, \boldsymbol{\Omega}] - \nu\operatorname{rot}\boldsymbol{v}\Big),$$

whence follows the expression for the pressure (1.7).

Applying the rot operation to (1.5), we obtain

$$\Delta\boldsymbol{v} + \varkappa^2\boldsymbol{v} = 0. \tag{1.9}$$

Equation (1.9) has the simplest form in the projection onto the z axis:

$$\Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} + \varkappa^2\Big)w = 0. \tag{1.10}$$

Let us substitute (1.6) into (1.10). Then, the amplitude $w_*(r)$ of the axial velocity component should satisfy the Bessel equation

$$\frac{d^2 w_*}{dr^2} + \frac{1}{r}\,\frac{dw_*}{dr} + \Big(-\frac{m^2}{r^2} + \lambda^2\Big)w_* = 0,$$

where $\lambda^2 = \varkappa^2 - k^2$. We note that amplitudes of the radial velocity u_* and azimuthal velocity v_* are related to w_* as follows:

$$u_* = \frac{im\varkappa}{\lambda^2 r}\, w_* + \frac{ik}{\lambda^2}\,\frac{dw_*}{dr}, \qquad v_* = -\frac{mk}{\lambda^2 r}\, w_* - \frac{\varkappa}{\lambda^2}\,\frac{dw_*}{dr}. \tag{1.11}$$

Below, the subscript asterisk is omitted.

2. Effect of Viscosity on Gyroscopic Waves. We introduce the dimensionless variables: $r' = r/a$, $z' = z/a$, $\tau = \omega/\Omega$, $\delta = b/a$, $\varkappa' = \varkappa a$, $k' = ka$, $\lambda' = \lambda a$, and the Ekman number $\mathrm{E} = \nu/(\Omega a^2)$. Below, the primes are omitted. Equation (1.8) in dimensionless variables becomes

$$\tau\varkappa - 2k - i\mathrm{E}\varkappa^3 = 0. \tag{2.1}$$

One of the roots of Eq. (2.1) depends on the parameters of large-scale motion of the liquid:

$$\varkappa_1 = -\frac{1}{2}\Big(\frac{s}{3\mathrm{E}} - i\,\frac{\tau}{s}\Big) - \frac{i\sqrt{3}}{2}\Big(\frac{s}{3\mathrm{E}} + i\,\frac{\tau}{s}\Big), \tag{2.2}$$

and the other two roots depend on the parameters of the boundary layers:

$$\varkappa_2 = \frac{s}{3\mathrm{E}} - i\,\frac{\tau}{s}, \qquad \varkappa_3 = -\frac{1}{2}\Big(\frac{s}{3\mathrm{E}} - i\,\frac{\tau}{s}\Big) + \frac{i\sqrt{3}}{2}\Big(\frac{s}{3\mathrm{E}} + i\,\frac{\tau}{s}\Big).$$

Here $s = (27k + 3\sqrt{3it^3\mathrm{E}^{-1} + 81k^2}\,)^{1/3}\mathrm{E}^{2/3}(\sqrt{3}/2 + i/2)$. In many cases, the Ekman number E is very small, and if the value of τ is not close to zero, we have

$$\varkappa_1 \approx \frac{2}{\tau}\,k + i\,\frac{8\mathrm{E}}{\tau^4}\,k^3 + O(\mathrm{E}^2), \qquad \varkappa_j \approx (-1)^{j+1}\sqrt{\frac{|\tau|}{2\mathrm{E}}}\Big(-\frac{\tau}{|\tau|} + i\Big) - \frac{k}{\tau} + O(\mathrm{E}) \quad (j = 2, 3).$$

Using (1.11), we seek a solution only for the axial velocity component w. This solution can be represented as

$$w = \sum_{l=1}^{3}\sum_{j=1}^{3} C_{lj} Z_m^{(l-\delta_{1j})}(\lambda_j r),$$

where C_{lj} are constants such that $C_{12} = C_{13} = C_{31} = 0$, δ_{1j} is a Kronecker delta function, which is equal to unity for $j = 1$, $Z_m^{(0)}(\lambda_1 r) = J_m(\lambda_1 r)$, $Z_m^{(1)}(\lambda_1 r) = Y_m(\lambda_1 r)$, $J_m(\lambda_1 r)$ and $Y_m(\lambda_1 r)$ are Bessel functions, $Z_m^{(l)}(\lambda_j r) = H_m^{(l-1)}(\lambda_j r)$ at $l = 2, 3$, $j = 2, 3$, $H_m^{(l-1)}(\lambda_j r)$ is a Hankel function of the mth order, and $\lambda_j^2 = \varkappa_j^2 - k^2$. It should be noted that, along with Bessel functions of the 1st and 2nd kinds, Eq. (2.3) contains Hankel functions, which are more convenient for describing boundary layers than the other cylindrical functions. The function $H_m^{(1)}(\lambda_j r)$ vanishes for an infinite value of the complex argument for which its imaginary part is positive, $H_m^{(2)}(\lambda_j r)$ vanishes when the imaginary part of the argument is negative. Substituting solution (2.2) into boundary conditions (1.3) and (1.4), transformed using (1.11) in such a manner that they contain only the axial velocity component, we obtain

$$\sum_{l=1}^{3}\sum_{j=1}^{3}\Big(\frac{m(\varkappa_j - k)}{\lambda_j^2}\, Z_m^{(l-\delta_{1j})}(\lambda_j) + \frac{k}{\lambda_j}\, Z_{m-1}^{(l-\delta_{1j})}(\lambda_j)\Big) C_{lj} = 0,$$

$$\sum_{l=1}^{3}\sum_{j=1}^{3}\Big(\frac{m(\varkappa_j - k)}{\lambda_j^2}\, Z_m^{(l-\delta_{1j})}(\lambda_j) - \frac{\varkappa_j}{\lambda_j}\, Z_{m-1}^{(l-\delta_{1j})}(\lambda_j)\Big) C_{lj} = 0,$$

$$\sum_{l=1}^{3}\sum_{j=1}^{3} Z_m^{(l-\delta_{1j})}(\lambda_j) C_{lj} = 0,$$

$$\sum_{l=1}^{3}\sum_{j=1}^{3}\Big(\frac{2m(\varkappa_j - mk - k) - \varkappa_j\lambda_j^2\delta^2}{\lambda_j^2}\, Z_m^{(l-\delta_{1j})}(\lambda_j\delta) - \frac{2(\varkappa_j - mk)}{\lambda_j}\, Z_{m-1}^{(l-\delta_{1j})}(\lambda_j\delta)\Big) C_{lj} = 0,$$

$$\sum_{l=1}^{3}\sum_{j=1}^{3}\Big(\frac{m(\varkappa_j^2 - 2k^2 + \varkappa_j k)}{\lambda_j^2}\, Z_m^{(l-\delta_{1j})}(\lambda_j\delta) - \frac{(\varkappa_j^2 - 2k^2)\delta}{\lambda_j}\, Z_{m-1}^{(l-\delta_{1j})}(\lambda_j\delta)\Big) C_{lj} = 0,$$

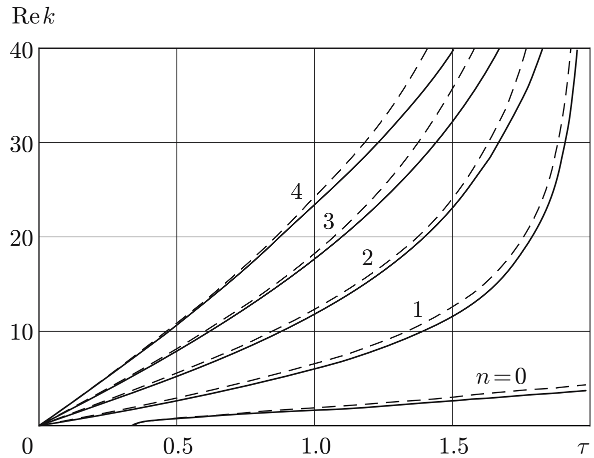

Fig. 1

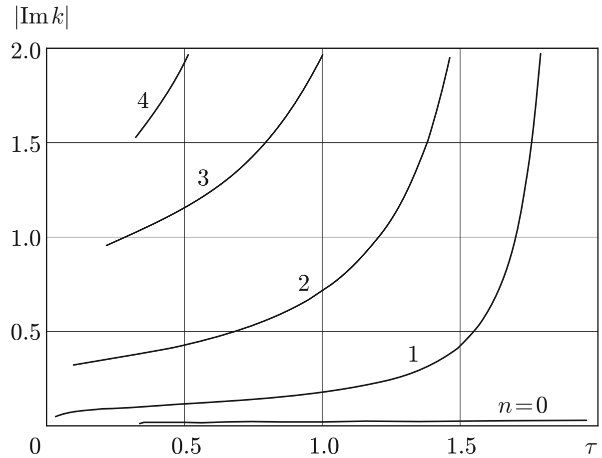

Fig. 2

Fig. 1. Real part of the wavenumber for the first four modes for $\mathrm{E} = 0.5 \cdot 10^{-4}$, $m = 1$, $\delta = 0.7$: the solid curves refer to $\mu \neq 0$ and the dashed curves to $\mu = 0$.

Fig. 2. Imaginary part of the wavenumber for the first four modes for $\mathrm{E} = 0.5 \cdot 10^{-4}$, $m = 1$, and $\delta = 0.7$.

$$\sum_{l=1}^{3}\sum_{j=1}^{3}\Big[\Big(\frac{2}{\varkappa_j} - \frac{m(\varkappa_j - k)}{\lambda_j^2\tau} - \frac{2im\mathrm{E}(m^2\varkappa_j - m^2k + m\varkappa_j - k + k\lambda_j^2\delta^2)}{\lambda_j^2\delta^2}\Big)Z_m^{(l-\delta_{1j})}(\lambda_j\delta)$$

$$+\Big(-\frac{k\delta}{\lambda_j\tau} + \frac{2i\mathrm{E}(m\varkappa_j - k)}{\lambda_j\delta}\Big)Z_{m-1}^{(l-\delta_{1j})}(\lambda_j\delta)\Big]C_{lj} = 0.$$

From the condition of solvability of this system of equations for the six unknown constants C_{lj}, one can derive the dispersion equation. It should be noted that the matrix is ill-conditioned; therefore, in the calculations, it is reasonable to redefine some of the coefficients C_{lj} as follows:

$$C'_{22} = C_{22}H_m^{(1)}(\lambda_2\delta), \quad C'_{23} = C_{23}H_m^{(1)}(\lambda_3\delta), \quad C'_{32} = C_{32}H_m^{(2)}(\lambda_2), \quad C'_{33} = C_{33}H_m^{(2)}(\lambda_3).$$

The dispersion curve has a countable set of branches. Figures 1 and 2 give the real and imaginary parts of the wavenumber k for the first four modes for $\mathrm{E} = 0.5 \cdot 10^{-4}$, $m = 1$, and $\delta = 0.7$. Surface waves correspond to $n = 0$. For comparison, Fig. 1 gives the dispersion curves obtained for the limiting case ($\mu = 0$), where the dispersion equation has the relatively simple form

$$[|k|\gamma J_{m-1}(|k|\gamma) - m(1 - 2/\tau)J_m(|k|\gamma)][|k|\delta\gamma Y_{m-1}(|k|\delta\gamma) - (m - 2m/\tau + 4 - \tau^2)Y_m(|k|\delta\gamma)]$$

$$- [|k|\delta\gamma J_{m-1}(|k|\delta\gamma) - (m - 2m/\tau + 4 - \tau^2)J_m(|k|\delta\gamma)]$$

$$\times [|k|\gamma Y_{m-1}(|k|\gamma) - m(1 - 2/\tau)Y_m(|k|\gamma)] = 0 \qquad (\gamma = \sqrt{4/\tau^2 - 1}\,).$$

The larger the mode number, the greater the effect of viscosity on the damping coefficient (the imaginary part of the wavenumber k) since an increase in the number leads to a complication of the spatial structure of the mode, an increase in the viscous stress, and an increase in the wave-energy dissipation.

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